

A SPECIAL SUBGROUP OF THE SURFACE BRAID GROUP

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1. INTRODUCTION

Herein we prove that if M is a compact oriented Riemann surface of genus g , and $M^{[n]}$ is the classifying space of n distinct, unordered points on M , then the kernel of the map $\pi_1(M^{[n]}) \rightarrow H_1(M)$ is generated by transpositions for sufficiently large n . Specifically we prove the following theorem:

Theorem 1. *If M is a polyhedron of genus g with n faces such that no face is a neighbor of itself and no two faces share more than one edge, then $B_n^0 = \ker(\pi_1(M^{[n]}) \rightarrow H_1(M))$ is generated by the edge set. Explicitly, the basepoint of $M^{[n]}$ may be chosen to be a marked point in the interior of each face, and each edge may be viewed as a transposition of the marked points on the faces it separates.*

2. DEFINITIONS

This section contains most of the pertinent definitions. Henceforth, M will be a polyhedron of genus g with n faces such that no face is a neighbor of itself and no two faces share more than one edge. If S is any orientable surface, let S^n be the subset of $S^{\times n}$ with all points distinct. Fadell and Neuwirth have shown [3] that forgetting a coordinate gives a fibration $S^n \rightarrow S^{n-1}$ with fiber homeomorphic to $S \setminus \{s_1, \dots, s_{n-1}\}$, where the s_j are distinct points in S . We will at some point use the homotopy extension and lifting principle of this map.

Let the set $\{f_j\}$ be the set of faces of M , E its edges, and V its vertices. Choose a basepoint x_j for each face f_j . Let $X = \{x_j\}$. The edge set, E , of M defines a graph Γ , and we have the dual graph $\tilde{\Gamma}$ with edge set E and vertices X . To each edge $e \in E$ viewed as an edge of $\tilde{\Gamma}$, we attach an orientation. Thus any curve crossing $e \in \Gamma$ crosses with sign relative to $e \in \tilde{\Gamma}$. We choose X as the basepoint for all homotopy groups.

Definition 2. *Let $x, y \in X$, and let $U \subset M$ open be such that $U \cap X = \{x, y\}$ and U is simply connected. When we say **transposition** in $\pi_1(M^{[n]}, X)$ of x and y , we mean a generator of $\pi_1(U^{[2]}, \{x, y\})$ extended by constants.*

The groups in question are as follows: $\pi_1(M^{[n]})$ is the surface braid group. $S^1 = \mathbb{R}/\mathbb{Z}$. $A : \pi_1(M^{[n]}) \rightarrow H_1(M)$ is the map which takes a curve $\gamma : S^1 \rightarrow M^{[n]}$, $\gamma = \{\gamma_1, \dots, \gamma_n\}$ to the union of the images $\gamma_j([0, 1])$. This gives a collection of closed curves without boundary. $B_n^0 = \ker(A : \pi_1(M^{[n]}) \rightarrow H_1(M))$. If $e \in E$, $\partial e = x_j - x_k$, then a transposition of x_j and x_k across e maps under A to $e - e = 0$. Let \tilde{B}_n^0 be the subgroup of B_n^0 generated by the transpositions associated to E . Clearly we intend to show that $\tilde{B}_n^0 = B_n^0$. For any surface N we let $\Pi_1(N)$ be the fundamental groupoid of N .

If $\gamma : ([0, 1], \{0, 1\}) \rightarrow (M^{[n]}, X)$ is a based map, we label $\gamma = \{\gamma_1, \dots, \gamma_n\}$ by $\gamma_j(0) = x_j$. We use $\gamma \mapsto [\gamma]$ to denote the map from based curves to $\pi_1(M^{[n]})$.

Definition 3. Let $\gamma = \{\gamma_1, \dots, \gamma_n\}$ be a curve in $M^{[n]}$ with $\gamma(0) = \gamma(1) = X$, which avoids the set V . Assume that γ intersects E for only finitely many t , and at each intersection, $\gamma_j(t) \in E$, $\gamma_j(t - \varepsilon)$ and $\gamma_j(t + \varepsilon)$ are contained in different faces for small ε . We call such a γ **enumerable**.

One can show that for any $b \in B_n^0$, there is an enumerable γ such that $[\gamma] = b$.

Definition 4. If $\gamma = \{\gamma_1, \dots, \gamma_n\}$ is an enumerable curve in $M^{[n]}$, the **edge set of γ_j** is the ordered set of edges $E_{\gamma_j} = (e_{j1}^\pm \dots e_{jp_j}^\pm)$ which are crossed by γ_j . The sign is positive if γ_j crosses $e_\alpha \in \Gamma$ in the same orientation as $e_\alpha \in \check{\Gamma}$. Notice that the image of γ_j in $\Pi_1(M \setminus V) = \Pi_1(\check{\Gamma})$ is the curve $e_{jp_1}^\pm \dots e_{j1}^\pm$. Define the **edge set of γ** as the collection of edge sets of its constituent curves: $E_\gamma = \{E_{\gamma_j}\}$. We will write $E_j = E_{\gamma_j}$.

The edge set E_j is the signed sequence of edges encountered by γ_j .

Definition 5. If $\gamma = \{\gamma_1, \dots, \gamma_n\}$ is an enumerable curve in $M^{[n]}$, and $E_{\gamma_j} = (e_{j1}^\pm \dots e_{jp_j}^\pm)$ is the edge set of γ_j , then the **face set of γ_j** is the set $X_{\gamma_j} = (x_{j0} \dots x_{jp_j})$ such that e_{jk}^\pm is the edge from $x_{j,k-1}$ to x_{jk} . Specifically, $x_{j0} = \gamma(0)$, $x_{jp_j} = \gamma(1)$.¹

The face set X_j is the sequence of faces encountered by γ_j .

Definition 6. We say that an edge set $X_j = (x_{j0} \dots x_{jp_j})$ is a **palindrome** if p_j is even and $x_{jk} = x_{j(p_j-k)}$ for all k . We will say that an edge set is a **palindrome** if its corresponding face set is a palindrome.

¹Notice that the edge and face sets are ordered left to right, but multiplication of curves is right to left.

Definition 7. We say that an enumerable curve γ is **balanced** if for every edge e_α , the multiplicity of e_α^+ in E_γ is equal to the multiplicity of e_α^- .

We are interested in balanced curves for the following reason:

Theorem 8. For any element $b \in B_n^0$, there is some balanced curve γ such that $[\gamma] = b$.

Proof. Choose any enumerable γ' with $[\gamma'] = b$. γ' need not be balanced, but since $b \in B_n^0$, the edge set of γ' is exact (a boundary of the homology complex of $\check{\Gamma}$: $0 \rightarrow \mathbb{Z}V \rightarrow \mathbb{Z}E \rightarrow \mathbb{Z}X \rightarrow 0$). However, for any vertex v (face of $\check{\Gamma}$), there are clearly curves γ'' such that $[\gamma''] = 1$, and $E_{\gamma''} = \partial v$. For example, if $\gamma_j'' \equiv x_j$ for $j \geq 2$, and γ_1'' travels from x_1 to v , once around v , and returns along the original path to x_1 , then $A\gamma'' = \pm \partial v$. Thus we may choose γ'' such that $[\gamma''] = 1$ and $E_{\gamma''} = E_{\gamma'}$, so that $\gamma = (\gamma'')^{-1}\gamma'$ is the desired curve. \square

3. ONE PARTICLE MOTION

In this section, we endeavor to prove that all one particle motion in B_n^0 lies in \tilde{B}_n^0 . First we will prove the following theorem.

Theorem 9. Let $\gamma = \{\gamma_1, \dots, \gamma_n\} : S^1 \rightarrow (M^{[n]}, X)$ such that γ_j is constant for $j > 1$ and X_1 is a palindrome, $(x_{j_1}x_{j_2} \dots x_{j_{m-1}}x_{j_m}x_{j_{m-1}} \dots x_{j_2}x_{j_1})$. Then $[\gamma] \in \tilde{B}_n^0$.

We prove this theorem by induction on m . We call m the **height** of γ .

Lemma 10. If γ is as in Theorem 9 and for some $k < m$, $x_{j_{k-1}} = x_{j_{k+1}}$, then there exist γ' , γ'' , and γ''' each with shorter palindromic face sets such that $[\gamma] = [\gamma''][\gamma'''][\gamma']$.

Proof. We give two constructions for this proof. Assume that for some $k < m$, $x_{j_{k-1}} = x_{j_{k+1}}$.

Construct γ' which follows γ until reaching x_{j_k} , wrapping an appropriate number of times around x_{j_m} , then retracing its path to x_1 . Likewise construct $(\gamma'')^{-1}$ following γ^{-1} in the same way (recall that E_1 is a palindrome, though γ and γ^{-1} need not behave in the same way at each vertex). Now, there is some curve in the homotopy class $[\gamma'']^{-1}[\gamma][\gamma']^{-1}$ with only its first coordinate non-constant and such that its face set is X_1 with $x_{j_{k-1}}$ and x_{j_k} removed. Let this be γ''' .

An alternative way to see this is as follows. If x_p and x_q are neighbors, then at any time γ_1 travels through x_p , a homotopic curve may be obtained by following the same path from x_1 to x_p , then travelling to x_q , returning to x_p , and completing γ . The effect this has on the face set is $(\dots x_p \dots) \mapsto (\dots x_p x_q x_p \dots)$. Beginning at $p = j_{k+1}$ we see that $[\gamma]$ may be realized by a curve with

$$E_1 = (x_{j_1} \dots x_{j_{k-1}} x_{j_k} x_{j_{k-1}} \dots x_{j_2} x_{j_1} x_{j_2} \dots x_{j_{k-2}} x_{j_{k+1}} \dots x_{j_m} \dots x_{j_1})$$

Since this curve returns to x_1 , one may factor it into two curves, the first being γ' . Factoring γ'' from the end of γ in the same way, we arrive at the curve γ''' . \square

Proof of Theorem 9. For simplicity we will assume that $j_1 = 1$, $j_2 = 2$, and that e_α connects x_1 to x_2 . Let $[\sigma_\alpha]$ be either transposition of x_1 and x_2 . We will write σ_α for some specific curve in $M^{[n]}$ fixing x_j , $j > 2$ and $[\sigma_\alpha]$ for the element in B_n^0 . σ_α^2 is homotopy equivalent to the curve in which γ_1 travels from x_2 , once around x_1 and back to x_2 , $\gamma_2 \equiv x_1$, and $\gamma_j \equiv x_j$, $j > 2$.

$\sigma_\alpha \gamma \sigma_\alpha^{-1}$ is a curve in which γ_1 travels from x_1 across e_α to x_2 , waits, then returns along e_α to x_1 . Thus γ_1 follows a homotopically trivial path, while γ_j remain constant for $j > 2$. Using the Fadell-Neuwirth fibration $\check{\Gamma}^n \rightarrow \check{\Gamma}^{n-1}$, forgetting the coordinate of x_2 , we may lift this homotopy to get a curve γ' with $[\gamma'] = [\sigma_\alpha][\gamma][\sigma_\alpha]^{-1}$, such that γ_j is constant for $j > 1$ and $\gamma_1(0) = x_2$.

We will construct γ' explicitly. Consider the map, $\gamma_1 : [0, 1] \rightarrow \check{\Gamma}$. We have realized this curve as a curve which travels from vertex to vertex, turning some number of times around each vertex. Lifting the homotopy has the effect of dragging the vertex x_2 to x_1 . Thus every time γ_1 reaches x_2 aside from the first and last, γ'_1 reaches x_2 travels to x_1 and around some number of times, then returns to x_2 .

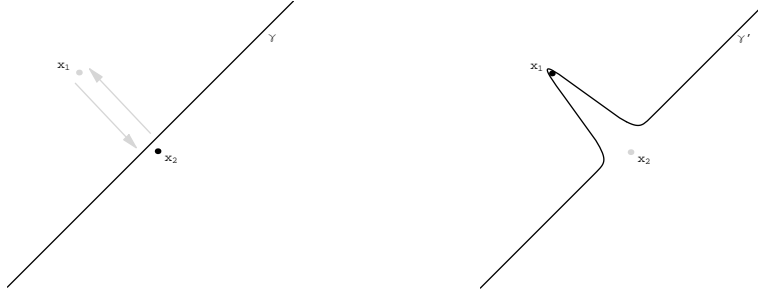


Figure 1. γ becomes γ' under conjugation.

Notice that there is some $\varepsilon > 0$ such that $\gamma_1|_{[\varepsilon, 1-\varepsilon]}$ is a curve in $\hat{\Gamma} \setminus \{x_j | j \neq 1\}$ while γ'_1 is a curve in $\hat{\Gamma} \setminus \{x_j | j \neq 2\}$. Thus we have drawn x_1 or x_2 in light gray in our figures as points that γ_1 or γ'_1 may cross without affecting its homotopy class.

By construction, $E_{\gamma'_1} = (e_\alpha^+ e_\alpha^- \cdots e_\alpha^+ e_\alpha^-)$. Therefore γ'_1 begins and ends by circling x_1 some number of times and returning to x_2 . This means that there is some γ'' and integers a and b such that

$$[\gamma''] = [\sigma_\alpha]^{2b} [\gamma'] [\sigma_\alpha]^{2a} = [\sigma_\alpha]^{2b+1} [\gamma] [\sigma_\alpha]^{2a-1},$$

and that $E_{\gamma''}$ is the same as $E_{\gamma'_1}$ without the beginning and ending copies of $e_\alpha^+ e_\alpha^-$. Left and right multiplication of γ by odd powers of σ ($L_{\sigma^{2b+1}} \circ R_{\sigma^{2a-1}}$) modifies the face set. It acts by taking each x_2 to $x_2 x_1 x_2$ (as in Figure 1). This is clear because this conjugation is the homotopy lifting map $\pi_1(\check{\Gamma} \setminus \{\hat{x}_1, x_2, \dots, x_n\}) \rightarrow \pi_1(\check{\Gamma} \setminus \{x_1, \hat{x}_2, \dots, x_n\})$ dragging x_2 along e_α^- to x_1 .

We have constructed a new curve, γ'' such that $X_{\gamma''}$ is similar to X_{γ_1} except that the first and last faces, which were both x_1 , have been removed, so that the first and last faces are now x_2 and all other instances in E_{γ_1} of x_2 have been replaced by the string $x_2 x_1 x_2$. Also, γ''_j are constant for $j > 1$.

$$(x_1 x_2 \dots x_2 \dots x_2 x_1) \mapsto (x_2 \dots x_2 x_1 x_2 \dots x_2).$$

Notice that this new edge set is still a palindrome.

Now we may apply Lemma 10 to factor γ'' into a product of curves. Continue to use x_{j_k} to denote the faces associated to γ . For every $k > 2$ such that $j_k = 2$, $X_{\gamma''}$ contains a copy of the string $x_2 x_1 x_2$. When $k \neq m$, we apply Lemma 10 to remove this string from the left and right. Working from smallest k to largest, we remove palindromes of height k from left and right, replacing instances of $x_2 x_1 x_2$ in the middle palindrome with x_2 .

We thus write γ'' as a product of $2K + 1$ curves, where K is the number of $2 < k < m$ such that $j_k = 2$. For each such k , there are two curves of height k , and one final curve of height m or $m - 1$. This final curve is the middle curve, and has the same face set as γ , except removal of the first and last faces, and if $j_m = 2$, the middle face is replaced: $x_2 \mapsto x_2 x_1 x_2$. Thus if $j_m \neq 2$, then the final curve has height $m - 1$ and we are done by induction on m .

Assume that $j_m = 2$. $[\sigma_\alpha]^{2b+1} [\gamma] [\sigma_\alpha]^{2a-1}$ factors into a set of curves, but the central curve has height m . That is, after applying Lemma 10, we factor out curves

of smaller height, and the remaining curve has face set

$$X_1 = (x_{j_2}x_{j_3} \dots x_{j_m}x_1x_{j_m} \dots x_{j_3}x_{j_2}).$$

If $j_3 \neq 1$ then we are done as before by applying the algorithm to get a palindrome of height $m - 1$. In the case that $j_3 = 1$, apply the argument again to get a curve with edge set

$$E_1 = (x_{j_3} \dots x_{j_m}x_{j_1}x_{j_2}x_{j_1}x_{j_m} \dots x_{j_3}) = (x_1 \dots x_2x_1x_2x_1x_2 \dots x_1).$$

This again has height m . However, notice that part of this curve travels through $x_2x_1x_2$. This part of the curve is an element of $\Pi_1(f_1 \cup f_2 \setminus \{x_2\})$ based in f_2 . This is contractible to $f_2 \setminus \{x_2\}$, so we may reduce the face set by $x_2x_1x_2 \mapsto x_2$. That is, a homotopy of γ_1 may pass freely through x_1 , since γ_1 is a curve in $\tilde{\Gamma} \setminus \{x_j | j > 1\}$. When doing this, we create a new curve of smaller height which is still a palindrome. Thus the theorem is proved. \square

Theorem 11. *If $\gamma = \{\gamma_1, \dots, \gamma_n\} : S^1 \rightarrow (M^{[n]}, X)$ such that $[\gamma] \in B_n^0$, and $\gamma_j \equiv x_j$ for $j > 1$, then $\gamma \in \tilde{B}_n^0$.*

Proof. The face set X_1 is not necessarily a palindrome (or, in fact, finite). We need only factor some representative of $[\gamma]$ into curves for which the face sets are palindromes. Assume, without loss of generality, that γ has finite face set. If $\phi : \tilde{M} \rightarrow M$ is the universal cover of M , let \hat{M} be a finite sub-polyhedron with boundary in \tilde{M} which contains $\phi^*\gamma$. In fact, $\hat{\pi} = \pi_1(\hat{M} \setminus \phi^{-1}(\{x_2, \dots, x_n\}))$ maps to B_n^0 and $[\gamma]$ is in its image. However, it is well-known that $\hat{\pi}$ is generated by curves which travel from x_1 to x_j , pass around x_j , and return to x_1 . Such a curve is a palindrome. \square

4. MANY PARTICLE MOTION

Ultimately, we intend to show that $B_n^0 = \tilde{B}_n^0$. Since we may represent elements of B_n^0 by balanced curves, we will study equivalences of balanced curves modulo \tilde{B}_n^0 . The first such equivalence says that if any curve begins by crossing e_α^+ then e_α^- , then it is equivalent to a curve which had not moved at all.

Lemma 12. *If γ is a balanced curve in B_n^0 with edge set $\{E_j\}$, and $E_1 = (e_\alpha^+ e_\alpha^- e_{13} e_{14} \dots e_{1p_1})$, then there is some element $b \in \tilde{B}_n^0$ and balanced γ' such that $[\gamma] = [\gamma']b$ and the edge set of γ' is $\{E'_1, E_2, \dots, E_n\}$ with $E'_1 = (e_{13} e_{14} \dots e_{1p_1})$.*

Proof. Assume that $\gamma_1(t_0) = x_1$ and that $\gamma_1|_{[0, t_0]}$ passes through only the edges e_α^+ and e_α^- . Assume further that no other curve passes through x_1 . We may vary γ slightly to insure these assumptions. Construct a new curve γ' by $\gamma'_j = \gamma_j$ for $j > 1$ and

$$\gamma'_1(t) = \begin{cases} x_1 & t < t_0 \\ \gamma(t) & t > t_0. \end{cases}$$

Notice that γ' has the appropriate edge set. Consider the curve $\delta = (\gamma')^{-1}\gamma$ under the Fadell-Neuwirth fibration forgetting the first coordinate.² Clearly the image of this curve is trivial. Therefore there is some element of $[\gamma']^{-1}[\gamma]$ such that all but the first coordinate is constant. By Theorem 11 we are done. \square

The second equivalence says that within an equivalence class, we may move the first letter of any word in the edge set to the beginning of some other word in the edge set. e.g.

$$\{(play), (spies), (mexico), \dots\} \equiv \{(splay), (pies), (mexico), \dots\}$$

Lemma 13. *If γ is a balanced curve in B_n^0 with edge set $\{(e_{jk})\}$ and $e_{11} = e_\alpha$ is an edge between x_1 and x_2 , then there is some element $b \in \tilde{B}_n^0$ and balanced γ' such that $[\gamma] = [\gamma']b$ and the edge set of γ' is $\{(e_{12} \dots e_{1p_1}), (e_\alpha e_{21} e_{22} \dots e_{2p_2} \dots), E_3, \dots, E_n\}$.*

Proof. Prepend to γ the curve σ_α^{-1} so that $[\gamma] = [\gamma\sigma_\alpha^{-1}][\sigma_\alpha]$. $\gamma\sigma_\alpha^{-1}$ now has edge set $\{(e_\alpha^{-1} e_\alpha e_{12} \dots e_{1p_1}), (e_\alpha e_{21} e_{22} \dots e_{2p_2} \dots), E_3, \dots, E_n\}$, where E_j is as in γ for $j > 2$, since only if $\gamma_j(0) = x_1, x_2$ will the edge set change. Finally one applies Lemma 12 to $[\gamma\sigma_\alpha]$. \square

This obviously gives an equivalence of face sets:

Corollary 14. *In the situation of Lemma 13, if the face set of γ is $\{(xy \dots)(y \dots) \dots\}$, then the face set of γ' is identical except for the switch $\{(y \dots)(xy \dots) \dots\}$.*

Note that x must appear at the beginning of exactly one word, call it the k^{th} , and likewise with y . This theorem allows us to move x from the beginning of the k^{th} to the beginning of the word starting with y . We will refer to this procedure as “removing the first face from the k^{th} word.”

We are now ready to prove the main theorem.

²As $\gamma_2, \dots, \gamma_n$ are constant, there is no problem lifting from $\tilde{\Gamma}^{[n]}$ to $\tilde{\Gamma}^n$.

Theorem 15. *If M is a polyhedron of genus g with n faces such that no side is a neighbor of itself and no two sides share more than one edge, then $B_n^0 = \ker(\pi_1(M^{[n]}) \rightarrow H_1(M))$ is generated by the edge set. Specifically, the basepoint of $M^{[n]}$ may be chosen to be a marked point in the interior of each face, and each edge may be viewed as a transposition of the marked points on the faces it separates.*

Proof. We show that any balanced curve γ is equivalent modulo \tilde{B}_n^0 to a balanced curve γ' with a smaller face set. It is clear that the only curve with face set of size zero is the identity.

We will do this by repeatedly applying Corollary 14 to remove the first face from various face sets in γ until we reach the case $X_1 = (x_2x_1x_2)$ or $X_1 = (x_1x_jx_1x_2)$. At this point, we apply Lemma 12 to reduce to $X_1 = (x_2)$ or $X_1 = (x_1x_2)$ while fixing all other X_j . Thus the face set is smaller, and by induction, we are done. In the proof of this theorem, we abandon the caveat that X_j has first face x_j .

Let X_1 be some curve with a nontrivial face set. First, we repeatedly remove the first face from the first curve so that X_1 has two elements. Relabel the x_j so that $X_1 = (x_1x_2)$. Let e_α denote the edge connecting x_1 to x_2 .

Since $e_\alpha \in E_\gamma$ and γ is balanced, some edge set contains the pair x_2x_1 . Call this edge set X_2 . Notice that the first instance of x_1 in X_2 need not occur as $\dots x_2x_1 \dots$. Next, we remove the first face X_2 until its first face is x_1 . At this point, $X_1 = (x_jx_1x_2)$ and $X_2 = (x_1x_k \dots)$. If $j = k$, then removing the first face of X_2 forces $X_1 = (x_1x_jx_1x_2)$ and we are done. Otherwise, removing the first faces of X_2 then X_1 places us in the situation $X_1 = (x_1x_2)$ and $X_2 = (x_k \dots)$:

$$\begin{array}{lll} X_1 = (x_jx_1x_2) & X_1 = (x_jx_1x_2) & X_1 = (x_1x_2) \\ X_2 = (x_1x_k \dots) & \mapsto X_2 = (x_k \dots) & \mapsto X_2 = (x_k \dots) \\ X_? = (x_k \dots) & X_? = (x_1x_k \dots) & X_? = (x_jx_1x_k \dots) \end{array}$$

In this process, we have reduced the number of times x_1 appears in X_2 . If we repeat this process exactly, eventually the first instance of x_1 in X_2 will be either $x_jx_1x_j$ or x_2x_1 . Then we are done. \square

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